Special finiteness (§4) & log terminal models (§5)

Recall: $\pi : X \to U$ projective

$n = \dim X$

Thm A_n Existence in pl flips

$f : X \to Z$ pl flipping cont (X,Δ) plt

$K_X + \Delta$ flipping contraction, $S = \lfloor \Delta \rfloor$ integral

$-S$ is f-ample, then $f^+ : X^+ \to Z$ exists

Thm B_n Special finiteness

$(X, \Delta_0 = S + A + B_0)$ dlt A ample, $B_0 \geq 0$

$\lfloor \Delta_0 \rfloor = S$ integral

$V \subseteq WDiv_{\mathbb{R}}(X)$ f.d.

$$\mathcal{L}_{S+A}(V) = \left\{ \Delta = S + A + B \text{ s.t. } \begin{array}{l} (X,\Delta) \text{ is lc} \\ B \geq 0 \end{array} \right\}$$

fix some fin collection $\mathcal{C}$
suppose ϕ only contracts in $\mathcal{C}$

Then there are finitely many bir maps $\phi_i : X \dashrightarrow X_i$ s.t. for $\Delta \in \mathcal{L}_{S+A}(V)$ & any WLCM$(X/U, \Delta)$ that doesn't contract S $\phi : X \dashrightarrow Y$, then ϕ is isomorphic to ϕ_i in a nbhd of S

Thm C_n existence of log terminal models

if (X,Δ) is klt, Δ big

$K_X + \Delta \sim_{\mathbb{R},u} D \geq 0$, then (X,Δ)

has a log terminal model

Thm D_n non-vanishing

Thm E_n finiteness of models

Thm F_n finite generation

$$F_{n-1} \overset{\text{HM}}{\Longrightarrow} A_n, \quad E_{n-1} \Rightarrow B_n$$

(MMP_{n-1} for $\mathbb{R}$)

$$A_n + B_n \Rightarrow C_n \qquad B_n + C_n + D_{n-1} \Rightarrow D_n$$

$$C_n + D_n \Rightarrow E_n + F_n$$

Key observation: MMP with scaling

finiteness $\Longleftrightarrow$ termination

Recognition principle for WLCM/LJM:

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] & Y \\ W \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

1) bir contraction

2) $\Gamma = \phi_* \Delta$

3) $K_Y + \Gamma$ nef

$f^*(K_X + \Delta) - g^*(K_Y + \Gamma)$ effective exceptional

$\Longrightarrow$ (Y, Γ) is a WLCM$(X/U, \Delta)$

Special finiteness

Thm $E_{n-1} \Longrightarrow B_n$

Pf: Suppose not, then $\exists$ $\phi_i : X \dashrightarrow Y_i$ & $\Delta_i \in \mathcal{L}_{S+A}(V)$ s.t. ϕ_i is WLCM$(X/U, \Delta_i)$ & doesn't contract S
+ div $\in \mathcal{C}$

s.t. if S_i = transform of S

$S_i \subseteq Y_i$ $\quad f_{ij} : Y_i \to Y_j$

is not iso in a nbhd of S_i

wlog, all the f_{ij} are small & S is integral

up to taking a subsequence,

$$\Delta_i \longrightarrow \Delta_\infty \in \mathcal{L}_{S+A}$$

& $\mathrm{Supp}(\Delta_i) = \mathrm{Supp}(\Delta_\infty)$

diophantine approximation

$$\rightsquigarrow \Delta_\infty \in \mathcal{L}^{\mathrm{int}}_{S+A}$$

& (X, Δ_i) are plt

& ϕ_i ample models

i.e. $K_{Y_i} + \Gamma_i$ $\quad \phi_{i*}\Delta_i =: \Gamma_i$

is ample

$\phi_i = \phi_j \iff \Delta_i = \Delta_j$

pick $\Delta_i \leq \Delta$ s.t. (X,Δ) plt so that

Pass to a log resolution

(X, Δ) is log smooth & (S, Θ) is terminal

X smooth

$\mathrm{Supp}(\Delta)$ normal crossing

$$\Theta = (\Delta - S)|_S$$

$$(K_X + \Delta)|_S = K_S + \Theta$$

$C = A|_S \qquad \tau_i := \phi_i|_S : S \to S_i$

<u>Lemma 4.1</u> $\exists$ divisors $C \le \Xi_i \le \underbrace{(\Delta_i - S)}_{\Theta_i \le \Theta}|_S$

s.t. τ_i is an ample model for $K_S + \Xi_i$

WLCM

By E_{n-1}, there are finitely many choices for S_i, so up to passing to subsequence, $f_{ij}|_{S_i} : S_i \xrightarrow{\sim} S_j$

$$(K_{Y_i} + S_i)|_{S_i} = K_{S_i} + \Phi_i$$

$$(K_{Y_i} + \Gamma_i)|_{S_i} = K_{S_i} + \Psi_i$$

$$0 \le \Phi_i \le \Psi_i \le \tau_{i*}\Theta$$

Plt pairs $\quad \text{coeff}(\Phi_i) \in \{1 - \frac{1}{n}\}_{n \in \mathbb{Z}_{>0}}$

$\text{coeff}(\Theta) < 1$

$\Longrightarrow$ only finitely many Φ_i

similar argument to bound the coefficients of $(\phi_{i*} B)|_{S_i}$ where $B \subseteq_{\text{comp}} \mathrm{Supp}(\Delta_i - S)$

$\Longrightarrow$ finitely many $(\phi_{i*} B)|_{S_i}$

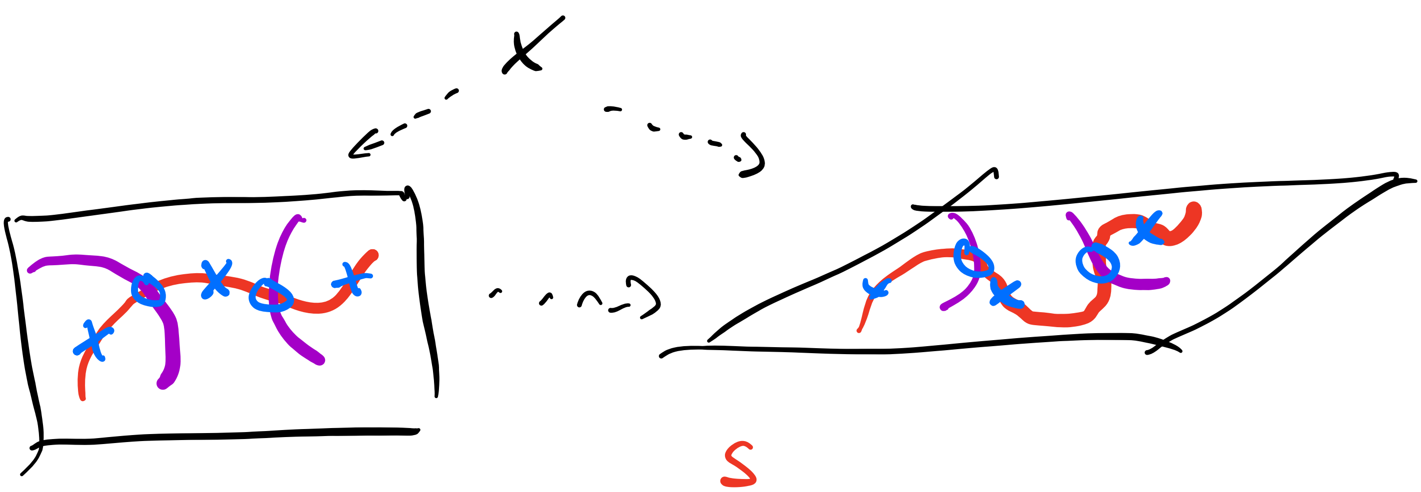
up to subsequence, suppose that

$$f_{ij}|_{S_i}^{*} \Phi_j = \Phi_i \qquad f_{ij}|_{S_i}^{*} (\phi_{j*} B)|_{S_j} = (\phi_{i*} B)|_{S_i}$$

Lemma 4.3

f_{ij} is an isomorphism in a nbhd of S_i

Pf



Lemma 4.1 $(X, S+A+B=\Delta)$ log smooth

$\lfloor \Delta \rfloor = S$

A ample $B \geq 0$

$\Theta = (\Delta - S)|_S$

$C = A|_S$ $\quad (S, \Theta)$ terminal

$\phi : X \dashrightarrow Y$ is a $WLCM(X/U, \Delta)$ doesn't contract S,

then the induced map $\tau = \phi|_S : S \dashrightarrow T$

is a $WLCM(S/U, \Xi)$ for a Ξ satisfying

$\tau_* \Xi = \Psi$ where $(K_Y + \Gamma)|_T = K_T + \Psi$.

Pf sketch

(X,Δ) plt $\Rightarrow$ (Y,Γ) plt $\Rightarrow$ (T,Ψ) klt

(S,Θ) terminal

$$\begin{tikzcd}
 & & R \arrow[dll, "f"'] \arrow[d] \arrow[drr, "g"] & & \\
W \arrow[d, "p"'] \arrow[drrrr, "q"] & & S \arrow[dll] \arrow[rr, dashed, "\tau"] & & T \arrow[d] \\
X \arrow[rrrr, dashed, "\phi"] & & & & Y
\end{tikzcd}$$

$$K_W + \Delta' = p^*(K_X + \Delta) + E$$

$$K_W + \Gamma' = q^*(K_Y + \Gamma) + F$$

$$K_R + \Theta' = f^*(K_S + \Theta) + E'$$

$$K_R + \Psi' = g^*(K_T + \Psi) + F'$$

$F \geq E \quad \Gamma' \leq \Delta'$

$F' \geq E' \quad \Psi' \leq \Theta'$

$\Longrightarrow$ τ doesn't extract divisors

$\Xi \leq \Theta$ maximal divisor s.t. $\tau_* \Xi = \Psi$, check via recognition principle

$$f^*(K_S + \Gamma) = g^*(K_T + \Upsilon) + L$$

need that L eff exceptional $\square$

<u>Lemma 4.3</u> (X, Δ_i) plt $S = \lfloor \Delta_i \rfloor$ integral

$\phi_i : X \dashrightarrow Y_i$ $\mathbb{Q}$-fact ample models of $K_X + \Delta_i$

ϕ_i doesn't contract S

$\phi_i|_S = \tau_i : S \dashrightarrow T_i$ $\qquad K_{T_i} + \Phi_i = (K_{Y_i} + T_i)|_{T_i}$

if 1) $f : Y_1 \dashrightarrow Y_2$ small

2) $f|_T = \sigma$ isomorphism

3) $\sigma^* \Phi_1 = \Phi_2$

4) for each comp B of $\operatorname{Supp}(\Delta_2 - S)$

$$(\phi_{1*} B)|_{T_1} = \sigma^*(\phi_{2*} B)|_{T_2}$$

$\Longrightarrow$ f is an isom in a nbhd of T_1.

Existence of log terminal models

<u>Thm</u> $A_n + B_n \Longrightarrow C_n$, that is,

assume existence of pl flips & special finiteness. Then for any klt pair (X,Δ) s.t. Δ big, $K_X+\Delta \sim_{\mathbb{R},u} D \geq 0$, $\exists$ a $K_X+\Delta$ minimal model.

Lemma 5.1 X is $\mathbb{Q}$ factorial

$\Delta = A+B+S$ S integral

$A, B, C \geq 0$

$B_+(A/u)$ doesn't contain the nklt centers of $(X, \Delta+C)$

images of divisors w/ log disc ≤ 0

Then any seq of flips & cont for $(K_X+\Delta)$-mmp w/ scaling by C which doesn't contract S is eventually disjoint from S

Pf sketch wlog S is irreducible

A general ample $f_i : X_i \dashrightarrow X_{i+1}$

S_i = transform S

the f_i contract finitely many divisors moreover, $\lambda_1, \lambda_2 \ldots \in [0,1]$

s.t. $\phi_i : X \dashrightarrow X_i$ is $WLCM(X/U, \Delta + \lambda_i C)$

apply special finiteness

$\Longrightarrow$ Finitely many ϕ_i in a nbhd of S

$+ \ \varepsilon \Longrightarrow f_i$ iso in a nbhd. $\square$

Lemma 5.2 Assume $A_u + B_{u}$, $(X, \Delta + C)$ as before + dlt

Suppose that $\exists$ D supported on $\operatorname{Supp}(S)$ s.t.

(*) $K_X + \Delta \sim_{\mathbb{R}, U} D + \alpha C \qquad \alpha \geq 0$

+ $K_X + \Delta + C$ is nef, then

$\exists$ a log terminal model $\phi : X \dashrightarrow Y$ s.t. $B_+(\phi_* A/U)$ doesn't contain nklt

Pf sketch run mmp with scaling

use (*) to see that every small

contraction is a pl flipping contraction + special termination holds.

Rmk - Def the output of the previous lemma satisfies $\phi : X \dashrightarrow Y$

1) ϕ bir contraction

2) ϕ only contracts components of D

3) Y $\mathbb{Q}$-factorial

4) $\mathbb{B}_+(\phi_* A/U)$ doesn't contain $\mathrm{nklt}(Y, \phi_* \Delta = \Gamma)$

5) dlt $(K_Y + \Gamma)$ nef

In this case we say ϕ is a nice model wrt D & A

Key Lemma (5.4 + 5.5)

$(X, \Delta = A + B)$ dlt pair

$A, B \geq 0$

s.t.

1) $K_X + \Delta \sim_{\mathbb{R}, U} D \geq 0$

2) $\mathrm{Supp}(\Delta + D) =: G$ suppose that (X, G) log smooth

3) $\mathbb{B}_+(A/U)$ doesn't contain any comp of $\mathrm{nklt}(X, G)$

then

a) Assuming $A_{n} + B_{n}$, (X, Δ) has a nice model wrt A, D

b) if every component of D is either semiample or contained in $\mathbb{B}(K_X + \Delta/U)$, then this nice model is an LTM.

PF a) $D = D_1 + D_2$ $\quad$ $D_1 \leq \lfloor \Delta \rfloor$

D_2 shares no comp w/ $\lfloor \Delta \rfloor$

if $D_2 = 0$, then it follows from Lemma 5.2 with $\lfloor \Delta \rfloor = S$

Induct on the components of D_2

$$c = \mathrm{lct}(X, \Delta ; D_2) > 0$$

$$D + cD_2 \qquad \Theta = \Delta + cD_2$$

$$K_X + \Theta \sim_{\mathbb{R}, U} D + cD_2$$

$\Longrightarrow$ $\lfloor \Theta \rfloor$ has more components & $D + cD_2$ has less components not contained in $\lfloor \Theta \rfloor$

induction $\Longrightarrow$ we have a nice model

b) Use the conditions to conclude that

$$p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + F$$

where

eff exceptional

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] & & Y \end{tikzcd}$$

log resolution + recognition principle ☒

Pf of thm $A_n + B_n \Rightarrow C_n$

Perturb so that $\Delta = A + B$

generic ample

effective

$$K_X + \Delta \sim_{\mathbb{R}, u} D \geq 0$$

write

$$D = M + F$$

very mobile

$\subseteq B(K_X + \Delta/u)$

resolve $f: Y \to X$ which resolves

base loci of <u>each</u> component of

M + Lemma 3.6.11 (prelim talk I)

$\exists$ Φ on Y s.t. $LTM(Y/U, \Phi)$

$K_Y + \Phi \sim_{\mathbb{R},U} N + G$ $\Leftarrow \subseteq B(K_Y + \Phi/U)$ $\quad \| \quad LTM(X/U, \Delta)$

semiample

& $\Phi + N + G$ normal crossings

by key lemma 5.4 + 5.5, $LTM(Y/U, \Phi)$

exists $\blacksquare$